Mathematical model for elastic beams with longitudinally variable depth

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Abstract. In this work we introduce a new mathematical model for elastic beams with a cross-section of constant width and longitudinally variable depth, obtained from the classical three-dimensional linear elasticity problem by using an asymptotic expansion method. We characterize the first- and second-order displacements and the first-order stress field, giving results related to existence, uniqueness and convergence for the limit model solution. Finally, we present the computations for a classical example.

Keywords: beam, longitudinally variable depth, asymptotic analysis, linear elasticity, limit model

1. Introduction

Asymptotic methods have been widely used for mathematical obtaining and justification of beam models in the framework of theory of elasticity during past two decades. First fundamental contribution in this direction was achieved by Bermúdez and Viaño [5] with the justification of the classical one-dimensional Bernoulli–Navier–Euler model for the bending of a linearized thermoelastic beam by adapting the asymptotic expansion method introduced by Ciarlet and Destuynder [7] for linearly elastic plates. The application of this method to different situations (linear and nonlinear elasticity, anisotropic and composite materials, static and dynamic cases and so on) has yielded important contributions. A complete survey of rod models with almost exhaustive bibliographic references may be found in Trabucho and Viaño [11].

During the last years, the authors (cf. [1,4]) have dealt with the case of beams with variable cross-section but remaining unchanged its principal axes of inertia respect to the reference axes. That means that the geometry of the cross-section depends on the longitudinal variable. Closely related to these beams are the ones, that we will call longitudinally variable cross-section beams, in which the dimensions in one direction (named width) remains constant, while the dimensions in the other direction (or depth) varies longitudinally. This kind of beams are widely used in civil engineeering, for instance, in bridges and frame structures. The aim of this paper is to obtain a mechanical model for longitudinally variable cross-section beams in the framework of linear elasticity.

In Section 2 we describe the physical problem, we state the mathematical model by an asymptotic analysis of the three-dimensional problem and we give several technical results. In next section we introduce the asymptotic development and we present the corresponding problem for each term of such expansion. We also obtain the limit problems that characterize the first- and second-order terms and we study existence, uniqueness and convergence of solution. Finally, Section 4 is devoted to the study of a particular beam with multiply connected cross-section.

2. Definition and modelling of a longitudinally variable cross-section beam

Let ε and L be positive real parameters representing, respectively, the maximum width of the cross-section and the length of the beam. Let $H \in W^{2,\infty}(0, L)$ be a "shape" function verifying:

$$\alpha \leqslant H(t) \leqslant 1, \quad \forall t \in [0, L] \text{ for some } \alpha \in (0, 1). \tag{1}$$

We consider the longitudinally variable cross-section elastic beam occupying the reference configuration $\overline{\Omega^\varepsilon}$ defined by $\Omega^\varepsilon = \Omega^{\varepsilon+} \cup \Omega^{\varepsilon-}$, where the fixed part $\Omega^{\varepsilon+}$ and the varying one $\Omega^{\varepsilon-}$ are given, respectively, by:

$$\Omega^{\varepsilon+} = \left\{(x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon)\colon -\varepsilon < x_1^\varepsilon < \varepsilon,\ 0 < x_2^\varepsilon < \frac{\varepsilon}{2},\ 0 < x_3^\varepsilon < L\right\},$$

$$\Omega^{\varepsilon-} = \left\{(x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon)\colon -\frac{\varepsilon}{2} < x_1^\varepsilon < \frac{\varepsilon}{2},\ -\varepsilon H(x_3^\varepsilon) < x_2^\varepsilon \leqslant 0,\ 0 < x_3^\varepsilon < L\right\}.$$

The different parts of the boundary $\partial\Omega^\varepsilon$ are given by:

$$\Gamma_0^\varepsilon = \{(x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) \in \partial\Omega^\varepsilon\colon x_3^\varepsilon = 0\}, \quad \Gamma_L^\varepsilon = \{(x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) \in \partial\Omega^\varepsilon\colon x_3^\varepsilon = L\},$$

$$\Gamma^{\varepsilon+} = \left\{(x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) \in \partial\Omega^\varepsilon\colon 0 \leqslant x_2^\varepsilon \leqslant \frac{\varepsilon}{2},\ 0 < x_3^\varepsilon < L\right\},$$

$$\Gamma_1^{\varepsilon-} = \left\{(x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) \in \partial\Omega^\varepsilon\colon x_1^\varepsilon = -\frac{\varepsilon}{2},\ -\varepsilon H(x_3^\varepsilon) < x_2^\varepsilon < 0,\ 0 < x_3^\varepsilon < L\right\},$$

$$\Gamma_2^{\varepsilon-} = \left\{(x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) \in \partial\Omega^\varepsilon\colon -\frac{\varepsilon}{2} \leqslant x_1^\varepsilon \leqslant \frac{\varepsilon}{2},\ x_2^\varepsilon = -\varepsilon H(x_3^\varepsilon),\ 0 < x_3^\varepsilon < L\right\},$$

$$\Gamma_3^{\varepsilon-} = \left\{(x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) \in \partial\Omega^\varepsilon\colon x_1^\varepsilon = \frac{\varepsilon}{2},\ -\varepsilon H(x_3^\varepsilon) < x_2^\varepsilon < 0,\ 0 < x_3^\varepsilon < L\right\}.$$

Thus, if we define $\Gamma^\varepsilon = \Gamma^{\varepsilon+} \cup \Gamma_1^{\varepsilon-} \cup \Gamma_2^{\varepsilon-} \cup \Gamma_3^{\varepsilon-}$, we have that $\partial\Omega^\varepsilon = \Gamma_0^\varepsilon \cup \Gamma_L^\varepsilon \cup \Gamma^\varepsilon$.

In Figs 1 and 2 we present the front and side views of two classical examples of beams with longitudinally variable depth: Fig. 1 corresponds to a linearly variable depth where the shape function H is given by:

$$H(x_3) = 1 - (1 - \alpha)\frac{x_3}{L}.$$

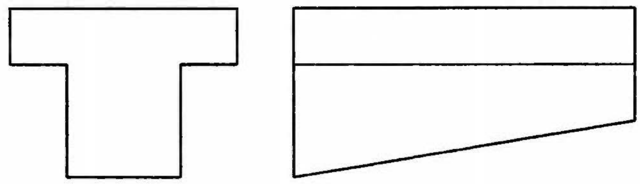

Fig. 1. Linearly variable depth beam.

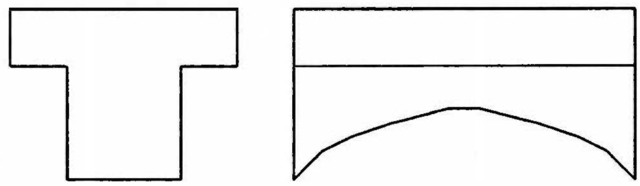

Fig. 2. Parabolically variable depth beam.

Fig. 2 represents a parabolically variable depth beam with:

$$H(x_3) = 1 - 4(1-\alpha)\left[\frac{x_3}{L} - \left(\frac{x_3}{L}\right)^2\right].$$

Remark 1. We remark that for cross-section of the beam we have chosen rectangular domains $\omega^{\varepsilon+} = (-\varepsilon, \varepsilon) \times (0, \varepsilon/2)$ and $\omega^{\varepsilon-} = (-\varepsilon/2, \varepsilon/2) \times (-\varepsilon, 0]$ for the sake of simplicity of notations and computations, but a general shape for $\omega^{\varepsilon+}$ and $\omega^{\varepsilon-}$ can be considered in the same way.

Remark 2. Condition $H \geqslant \alpha > 0$ in $[0, L]$ is inspired by mathematical reasons. However, we can also obtain a model for the important case where $H(t) = 0$ for some $t \in [0, L]$ (see Remark 8 below).

We denote by $n^\varepsilon = (n_i^\varepsilon)$ the outward normal vector to $\partial\Omega^\varepsilon$ and by $\partial_i^\varepsilon v$ the partial derivative $\partial v/\partial x_i^\varepsilon$. For functions only depending on variable x_3^ε, the derivative will be denoted by a prime. Here and along the whole work we use, as it is customary in mathematical elasticity theory, the summation convention on repeated indices, supposing the Latin indices range over $\{1, 2, 3\}$ and the Greek ones over $\{1, 2\}$.

Our aim is the study of the mechanical behavior of a longitudinally variable cross-section elastic beam, supposed to be clamped at both ends and submitted to a system of body and surface forces. We assume the beam to be made of an homogeneous isotropic material of Saint Venant–Kirchhoff's type with Young's modulus E and Poisson's ratio ν. Thus, in the linearized elasticity framework, the displacement field u^ε and the stress tensor σ^ε are the solution of the following problem (see, for instance, Ciarlet [6]):

$$-\partial_j^\varepsilon \sigma_{ij}^\varepsilon = f_i^\varepsilon \quad \text{in } \Omega^\varepsilon, \tag{2}$$

$$u^\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon \cup \Gamma_L^\varepsilon, \tag{3}$$

$$\sigma_{ij}^\varepsilon n_j^\varepsilon = g_i^\varepsilon \quad \text{on } \Gamma^\varepsilon, \tag{4}$$

where the stress tensor obeys Hooke's law:

$$\sigma_{ij}^\varepsilon = \frac{E}{1+\nu} e_{ij}^\varepsilon(u^\varepsilon) + \frac{E\nu}{(1+\nu)(1-2\nu)} e_{kk}^\varepsilon(u^\varepsilon)\delta_{ij}, \tag{5}$$

for $e^\varepsilon(u^\varepsilon) \equiv (e^\varepsilon_{ij}(u^\varepsilon))$ the linearized strain tensor given by:

$$e^\varepsilon_{ij}(u^\varepsilon) = \frac{1}{2}(\partial^\varepsilon_i u^\varepsilon_j + \partial^\varepsilon_j u^\varepsilon_i). \tag{6}$$

If we consider the following functional spaces:

$$V(\Omega^\varepsilon) = \{v^\varepsilon \equiv (v^\varepsilon_i) \in [H^1(\Omega^\varepsilon)]^3\colon\ v^\varepsilon_i = 0 \text{ on } \Gamma^\varepsilon_0 \cup \Gamma^\varepsilon_L\},$$
$$\Sigma(\Omega^\varepsilon) = \{\tau^\varepsilon \equiv (\tau^\varepsilon_{ij}) \in [L^2(\Omega^\varepsilon)]^9\colon\ \tau^\varepsilon_{ij} = \tau^\varepsilon_{ji}\}$$

we can reformulate the problem by using the Hellinger–Reissner mixed variational form:

$$(u^\varepsilon, \sigma^\varepsilon) \in V(\Omega^\varepsilon) \times \Sigma(\Omega^\varepsilon)\colon$$
$$\int_{\Omega^\varepsilon} \left(\frac{1+\nu}{E}\sigma^\varepsilon_{ij} - \frac{\nu}{E}\sigma^\varepsilon_{kk}\delta_{ij}\right)\tau^\varepsilon_{ij}\,\mathrm{d}x^\varepsilon - \int_{\Omega^\varepsilon} e^\varepsilon_{ij}(u^\varepsilon)\tau^\varepsilon_{ij}\,\mathrm{d}x^\varepsilon = 0, \quad \forall \tau^\varepsilon \in \Sigma(\Omega^\varepsilon), \tag{7}$$
$$\int_{\Omega^\varepsilon} \sigma^\varepsilon_{ij} e^\varepsilon_{ij}(v^\varepsilon)\,\mathrm{d}x^\varepsilon = \int_{\Omega^\varepsilon} f^\varepsilon_i v^\varepsilon_i\,\mathrm{d}x^\varepsilon + \int_{\Gamma^\varepsilon} g^\varepsilon_i v^\varepsilon_i\,\mathrm{d}a^\varepsilon, \quad \forall v^\varepsilon \in V(\Omega^\varepsilon). \tag{8}$$

Remark 3. The method that follows can be also applied, for instance, to the cantilever beam corresponding to the three-dimensional problem:

$$-\partial^\varepsilon_j \sigma^\varepsilon_{ij} = f^\varepsilon_i \quad \text{in } \Omega^\varepsilon,$$
$$u^\varepsilon = 0 \quad \text{on } \Gamma^\varepsilon_0,$$
$$\sigma^\varepsilon_{ij} n^\varepsilon_j = g^\varepsilon_i \quad \text{on } \Gamma^\varepsilon,$$
$$\sigma^\varepsilon_{i3} = p^\varepsilon_i \quad \text{on } \Gamma^\varepsilon_L$$

using the space:

$$V(\Omega^\varepsilon) = \{v^\varepsilon \equiv (v^\varepsilon_i) \in [H^1(\Omega^\varepsilon)]^3\colon\ v^\varepsilon_i = 0 \text{ on } \Gamma^\varepsilon_0\}$$

(cf. Trabucho and Viaño [10, Chapter V]).

In order to introduce a straight beam, we define the open set $\omega = \omega^+ \cup \omega^-$, where

$$\omega^+ = (-1, 1) \times \left(0, \frac{1}{2}\right), \qquad \omega^- = \left(-\frac{1}{2}, \frac{1}{2}\right) \times (-1, 0].$$

Thus, we consider the reference beam of constant cross-section ω occupying the volume $\overline{\Omega}$ given by:

$$\Omega = \omega \times (0, L),$$

and we define the different parts of the boundary by:

$$
\begin{aligned}
&\Gamma_0 = \overline{\omega} \times \{0\}, && \Gamma_L = \overline{\omega} \times \{L\},\\
&\Gamma^+ = \gamma^+ \times (0,L) && \text{with } \gamma^+ = \left\{(x_1,x_2) \in \partial\omega\colon\ 0 \leqslant x_2 \leqslant \frac{1}{2}\right\},\\
&\Gamma_1^- = \gamma_1^- \times (0,L) && \text{with } \gamma_1^- = \left\{(x_1,x_2) \in \partial\omega\colon\ x_1 = -\frac{1}{2},\ -1 < x_2 < 0\right\},\\
&\Gamma_2^- = \gamma_2^- \times (0,L) && \text{with } \gamma_2^- = \left\{(x_1,x_2) \in \partial\omega\colon\ -\frac{1}{2} \leqslant x_1 \leqslant \frac{1}{2},\ x_2 = -1\right\},\\
&\Gamma_3^- = \gamma_3^- \times (0,L) && \text{with } \gamma_3^- = \left\{(x_1,x_2) \in \partial\omega\colon\ x_1 = \frac{1}{2},\ -1 < x_2 < 0\right\},\\
&\Gamma = \gamma \times (0,L) && \text{with } \gamma = \gamma^+ \cup \gamma_1^- \cup \gamma_2^- \cup \gamma_3^-.
\end{aligned}
$$

Remark 4. Although, for the sake of simplicity, we have chosen the reference cross-section such that $|\omega^+| = |\omega^-| = 1$, we can take a general section:

$$\omega = (-a,a) \times (0,b) \cup (-c,c) \times (-d,0].$$

We introduce the change of variable from the fixed domain Ω to Ω^ε:

$$\pi^\varepsilon : x \equiv (x_1,x_2,x_3) \in \overline{\Omega} \to \pi^\varepsilon(x_1,x_2,x_3) = \big(\varepsilon x_1, \varepsilon x_2 h(x_3), x_3\big) \equiv (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) \equiv x^\varepsilon \in \overline{\Omega^\varepsilon}, \tag{9}$$

where function h is defined as:

$$h(x_3) = \begin{cases} 1 & \text{if } x_2 \geqslant 0,\\ H(x_3) & \text{otherwise.}\end{cases}$$

We must remark that, although h is not smooth, the function $\pi^\varepsilon \in [W^{1,\infty}(\Omega)]^3$. Thus, for each function $\Phi^\varepsilon : x^\varepsilon \in \overline{\Omega^\varepsilon} \to \Phi^\varepsilon(x^\varepsilon) \in R$ we denote by Φ the new function $\Phi : x \in \overline{\Omega} \to \Phi(x) \in R$ given by $\Phi = \Phi^\varepsilon \circ \pi^\varepsilon$, i.e.:

$$\Phi(x_1,x_2,x_3) = \Phi^\varepsilon\big(\varepsilon x_1, \varepsilon x_2 h(x_3), x_3\big).$$

This function verifies, in a trivial way, the following properties:

$$
\begin{aligned}
&\partial_1^\varepsilon \Phi^\varepsilon = \varepsilon^{-1}\partial_1\Phi, \quad \partial_2^\varepsilon \Phi^\varepsilon = \varepsilon^{-1}h^{-1}\partial_2\Phi, \quad \partial_3^\varepsilon \Phi^\varepsilon = \partial_3\Phi - x_2 h^{-1}h'\partial_2\Phi,\\
&\int_{\Omega^\varepsilon} \Phi^\varepsilon \, \mathrm{d}x^\varepsilon = \varepsilon^2 \int_\Omega h\Phi \, \mathrm{d}x,\\
&\int_{\Gamma^{\varepsilon+}} \Phi^\varepsilon \, \mathrm{d}a^\varepsilon = \varepsilon \int_{\Gamma^+} h\Phi \, \mathrm{d}a, \quad \int_{\Gamma_1^{\varepsilon-}} \Phi^\varepsilon \, \mathrm{d}a^\varepsilon = \varepsilon \int_{\Gamma_1^-} h\Phi \, \mathrm{d}a,\\
&\int_{\Gamma_2^{\varepsilon-}} \Phi^\varepsilon \, \mathrm{d}a^\varepsilon = \varepsilon \int_{\Gamma_2^-} h^*(\varepsilon)\Phi \, \mathrm{d}a, \quad \int_{\Gamma_3^{\varepsilon-}} \Phi^\varepsilon \, \mathrm{d}a^\varepsilon = \varepsilon \int_{\Gamma_3^-} h\Phi \, \mathrm{d}a,
\end{aligned}
$$

where

$$h^*(\varepsilon) = \sqrt{1+\varepsilon^2(h')^2} = 1 + \frac{1}{2}\varepsilon^2(h')^2 - \frac{1}{8}\varepsilon^4(h')^4 + \cdots.$$

Now we scale the different fields appearing in the variational formulation. So, we define the rescaled fields $u(\varepsilon)$ and $\sigma(\varepsilon)$ by:

$$u_\alpha(\varepsilon)(x) = \varepsilon u_\alpha^\varepsilon(x^\varepsilon), \quad u_3(\varepsilon)(x) = u_3^\varepsilon(x^\varepsilon), \tag{10}$$

$$\sigma_{\alpha\beta}(\varepsilon)(x) = \varepsilon^{-2}\sigma_{\alpha\beta}^\varepsilon(x^\varepsilon), \quad \sigma_{\alpha 3}(\varepsilon)(x) = \varepsilon^{-1}\sigma_{\alpha 3}^\varepsilon(x^\varepsilon), \quad \sigma_{33}(\varepsilon)(x) = \sigma_{33}^\varepsilon(x^\varepsilon). \tag{11}$$

We also assume that the system of applied forces is such that:

$$f_\alpha^\varepsilon(x^\varepsilon) = \varepsilon f_\alpha(x), \quad f_3^\varepsilon(x^\varepsilon) = f_3(x), \tag{12}$$

$$g_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 g_\alpha(x), \quad g_3^\varepsilon(x^\varepsilon) = \varepsilon g_3(x), \tag{13}$$

where $f_i \in L^2(\Omega)$ and $g_i \in L^2(\Gamma)$ are independent of ε.

Thus, if we define:

$$V(\Omega) = [W(\Omega)]^3 = \{v \equiv (v_i) \in [H^1(\Omega)]^3\colon v_i = 0 \text{ on } \Gamma_0 \cup \Gamma_L\},$$

$$\Sigma(\Omega) = [L^2(\Omega)]_S^9 = \{\tau \equiv (\tau_{ij}) \in [L^2(\Omega)]^9\colon \tau_{ij} = \tau_{ji}\},$$

we obtain that $(u(\varepsilon), \sigma(\varepsilon))$ is the only solution of the following scaled variational problem posed in Ω:

$$\begin{aligned}
&(u(\varepsilon), \sigma(\varepsilon)) \in V(\Omega) \times \Sigma(\Omega)\colon\\
&-\int_\Omega h e_{ij}^*(u(\varepsilon))\tau_{ij}\,\mathrm{d}x + \int_\Omega \frac{1}{E} h\sigma_{33}(\varepsilon)\tau_{33}\,\mathrm{d}x\\
&\quad + \varepsilon^2 \int_\Omega h\left\{2\frac{1+\nu}{E}\sigma_{\alpha 3}(\varepsilon)\tau_{\alpha 3} - \frac{\nu}{E}(\sigma_{33}(\varepsilon)\tau_{\alpha\alpha} + \sigma_{\alpha\alpha}(\varepsilon)\tau_{33})\right\}\mathrm{d}x\\
&\quad + \varepsilon^4 \int_\Omega h\left\{\frac{1+\nu}{E}\sigma_{\alpha\beta}(\varepsilon) - \frac{\nu}{E}\sigma_{\gamma\gamma}(\varepsilon)\delta_{\alpha\beta}\right\}\tau_{\alpha\beta}\,\mathrm{d}x = 0, \quad \forall \tau \in \Sigma(\Omega),
\end{aligned} \tag{14}$$

$$\begin{aligned}
\int_\Omega h\sigma_{ij}(\varepsilon)e_{ij}^*(v)\,\mathrm{d}x &= \int_\Omega h f_i v_i\,\mathrm{d}x + \int_{\Gamma^+\cup\Gamma_1^-\cup\Gamma_3^-} h g_i v_i\,\mathrm{d}a\\
&\quad + \int_{\Gamma_2^-} h^*(\varepsilon) g_i v_i\,\mathrm{d}a, \quad \forall v \in V(\Omega),
\end{aligned} \tag{15}$$

where $e^*(v) \equiv (e_{ij}^*(v))$ is the generalized strain tensor defined by:

$$e_{11}^*(v) = \partial_1 v_1, \tag{16}$$

$$e_{12}^*(v) = \frac{1}{2}[\partial_1 v_2 + h^{-1}\partial_2 v_1], \tag{17}$$

$$e_{22}^*(v) = h^{-1}\partial_2 v_2, \tag{18}$$

$$e_{13}^*(v) = \frac{1}{2}[\partial_1 v_3 + \partial_3 v_1 - x_2 h^{-1} h' \partial_2 v_1], \tag{19}$$

$$e_{23}^*(v) = \frac{1}{2}[h^{-1}\partial_2 v_3 + \partial_3 v_2 - x_2 h^{-1} h' \partial_2 v_2], \tag{20}$$

$$e_{33}^*(v) = \partial_3 v_3 - x_2 h^{-1} h' \partial_2 v_3. \tag{21}$$

3. Characterization of the limit problems

In order to study the scaled three-dimensional problem as the thickness ε tends to zero we assume the asymptotic expansion:

$$(u(\varepsilon), \sigma(\varepsilon)) = (u^0, \sigma^0) + \varepsilon^2(u^2, \sigma^2) + \varepsilon^4(u^4, \sigma^4) + \cdots. \tag{22}$$

We must note that in the expansion only even powers of ε appear, since the terms corresponding to odd powers are null (cf. [11]).

If we substitute this formal expression into the scaled variational problem (14), (15), we obtain that the first term of the asymptotic expansion (u^0, σ^0) must satisfy:

$$\int_\Omega \frac{1}{E} h\sigma_{33}^0 \tau_{33}\,\mathrm{d}x - \int_\Omega h e_{33}^*(u^0)\tau_{33}\,\mathrm{d}x = 0, \quad \forall \tau_{33} \in L^2(\Omega), \tag{23}$$

$$\int_\Omega h e_{\alpha 3}^*(u^0)\tau_{\alpha 3}\,\mathrm{d}x = 0, \quad \forall(\tau_{\alpha 3}) \in [L^2(\Omega)]^2, \tag{24}$$

$$\int_\Omega h e_{\alpha\beta}^*(u^0)\tau_{\alpha\beta}\,\mathrm{d}x = 0, \quad \forall(\tau_{\alpha\beta}) \in [L^2(\Omega)]_S^4, \tag{25}$$

$$\begin{aligned}&\int_\Omega h\sigma_{\alpha\beta}^0 e_{\alpha\beta}^*(v_\gamma)\,\mathrm{d}x + \int_\Omega 2h\sigma_{\alpha 3}^0 e_{\alpha 3}^*(v_\gamma)\,\mathrm{d}x \\ &\quad = \int_\Omega h f_\alpha v_\alpha\,\mathrm{d}x + \int_{\Gamma^+\cup\Gamma_1^-\cup\Gamma_3^-} h g_\alpha v_\alpha\,\mathrm{d}a + \int_{\Gamma_2^-} g_\alpha v_\alpha\,\mathrm{d}a, \quad \forall(v_\gamma) \in [W(\Omega)]^2,\end{aligned} \tag{26}$$

$$\begin{aligned}&\int_\Omega h\sigma_{33}^0 e_{33}^*(v_3)\,\mathrm{d}x + \int_\Omega 2h\sigma_{\alpha 3}^0 e_{\alpha 3}^*(v_3)\,\mathrm{d}x \\ &\quad = \int_\Omega h f_3 v_3\,\mathrm{d}x + \int_{\Gamma^+\cup\Gamma_1^-\cup\Gamma_3^-} h g_3 v_3\,\mathrm{d}a + \int_{\Gamma_2^-} g_3 v_3\,\mathrm{d}a, \quad \forall v_3 \in W(\Omega).\end{aligned} \tag{27}$$

In the same way, the second term (u^2, σ^2) must satisfy:

$$\int_\Omega \frac{1}{E} h\sigma_{33}^2 \tau_{33}\,\mathrm{d}x - \int_\Omega h e_{33}^*(u^2)\tau_{33}\,\mathrm{d}x = \int_\Omega \frac{\nu}{E} h\sigma_{\alpha\alpha}^0 \tau_{33}\,\mathrm{d}x, \quad \forall \tau_{33} \in L^2(\Omega), \tag{28}$$

$$\int_\Omega h e_{\alpha 3}^*(u^2)\tau_{\alpha 3}\,\mathrm{d}x = \int_\Omega \frac{2(1+\nu)}{E} h\sigma_{\alpha 3}^0 \tau_{\alpha 3}\,\mathrm{d}x, \quad \forall(\tau_{\alpha 3}) \in [L^2(\Omega)]^2, \tag{29}$$

$$\int_{\Omega} h e^*_{\alpha\beta}(u^2)\tau_{\alpha\beta}\,dx = -\int_{\Omega} \frac{\nu}{E} h\sigma^0_{33}\tau_{\alpha\alpha}\,dx, \quad \forall(\tau_{\alpha\beta}) \in [L^2(\Omega)]^4_S, \tag{30}$$

$$\int_{\Omega} h\sigma^2_{\alpha\beta} e^*_{\alpha\beta}(v_\gamma)\,dx + \int_{\Omega} 2h\sigma^2_{\alpha 3} e^*_{\alpha 3}(v_\gamma)\,dx = \int_{\Gamma_2^-} \frac{1}{2}(h')^2 g_\alpha v_\alpha\,da, \quad \forall(v_\gamma) \in [W(\Omega)]^2, \tag{31}$$

$$\int_{\Omega} h\sigma^2_{33} e^*_{33}(v_3)\,dx + \int_{\Omega} 2h\sigma^2_{\alpha 3} e^*_{\alpha 3}(v_3)\,dx = \int_{\Gamma_2^-} \frac{1}{2}(h')^2 g_3 v_3\,da, \quad \forall v_3 \in W(\Omega). \tag{32}$$

Similar expressions can be obtained for the following terms of the asymptotic expansion: (u^4,σ^4), (u^6,σ^6) and so on.

We are going to obtain now the solution of the limit problem. In order to do this we introduce the space of generalized Bernoulli–Navier displacements:

$$V^*_{BN}(\Omega) = \{v \equiv (v_i) \in V(\Omega)\colon\ e^*_{\alpha\beta}(v) = e^*_{\alpha 3}(v) = 0\}. \tag{33}$$

We have the following characterization result for this space:

Lemma 1. *The space $V^*_{BN}(\Omega)$ is given by:*

$$\begin{aligned} V^*_{BN}(\Omega) = \{v \equiv (v_i)\colon\ & v_\alpha(x_1,x_2,x_3) = \zeta_\alpha(x_3),\ \zeta_\alpha \in H^2_0(\omega), \\ & v_3(x_1,x_2,x_3) = \zeta_3(x_3) - x_1\zeta_1'(x_3) - x_2 h(x_3)\zeta_2'(x_3),\ \zeta_3 \in H^1_0(\omega)\}. \end{aligned}$$

Proof. Since $e^*_{\alpha\beta}(v) = 0$, we have that $\partial_1 v_1 = \partial_2 v_2 = \partial_1 v_2 + h^{-1}\partial_2 v_1 = 0$. So:

$$v_1 = \zeta_1 + x_2 h z, \quad v_2 = \zeta_2 - x_1 z, \quad \zeta_\alpha, z \in H^1_0(0,L). \tag{34}$$

From $e^*_{\alpha 3}(v) = 0$ we have:

$$\partial_1 v_3 + \partial_3 v_1 - x_2 h^{-1} h' \partial_2 v_1 = 0, \tag{35}$$

$$h^{-1}\partial_2 v_3 + \partial_3 v_2 = 0. \tag{36}$$

Substituting expression (34) into these equations we obtain $\partial_3 z = \partial_{\alpha\beta} v_3 = 0$. Since $z \in H^1_0(0,L)$, $\partial_3 z = 0$ implies that $z = 0$. Thus,

$$v_\alpha = \zeta_\alpha, \quad \zeta_\alpha \in H^1_0(0,L).$$

Finally, taking into account (35), (36), from $\partial_{\alpha\beta} v_3 = 0$ we can conclude:

$$v_3 = \zeta_3 - x_1\zeta_1' - x_2 h\zeta_2', \quad \zeta_3 \in H^1_0(0,L),\ \zeta_\alpha \in H^2_0(0,L). \qquad \square$$

Then, as a consequence of Eqs (23)–(27), we have the following characterization of the limit problem corresponding to u^0 and σ^0_{33}:

Theorem 1. *If the system of applied forces verifies:*

$$f_\alpha \in L^2(\Omega), \quad g_\alpha \in L^2(\Gamma), \quad f_3 \in H^1(0,L;L^2(\omega)), \quad g_3 \in H^1(0,L;L^2(\gamma))$$

then the limit displacement u^0 belongs to space $V^*_{BN}(\Omega)$, that is:

$$u^0_\alpha(x_1,x_2,x_3) = \xi_\alpha(x_3), \quad \xi_\alpha \in H^2_0(0,L), \tag{37}$$

$$u^0_3(x_1,x_2,x_3) = \xi_3(x_3) - x_1\xi_1'(x_3) - x_2h(x_3)\xi_2'(x_3), \quad \xi_3 \in H^1_0(0,L), \tag{38}$$

where ξ_i are solution of the coupled problem:

$$-\int_0^L E\Big(\int_\omega x_1^2 h\Big)\xi_1'' v_1'' = \int_0^L M_1 v_1' - \int_0^L F_1 v_1, \quad \forall v_1 \in H^2_0(0,L), \tag{39}$$

$$\int_0^L E\Big[\Big(\int_\omega x_2h^2\Big)\xi_3' - \Big(\int_\omega x_2^2h^3\Big)\xi_2''\Big]v_2'' = \int_0^L M_2 v_2' - \int_0^L F_2 v_2, \quad \forall v_2 \in H^2_0(0,L), \tag{40}$$

$$\int_0^L E\Big[\Big(\int_\omega h\Big)\xi_3' - \Big(\int_\omega x_2h^2\Big)\xi_2''\Big]v_3' = \int_0^L F_3 v_3, \quad \forall v_3 \in H^1_0(0,L), \tag{41}$$

with:

$$F_i = \int_\omega hf_i + \int_{\gamma^+\cup\gamma_1^-\cup\gamma_3^-} hg_i + \int_{\gamma_2^-} g_i, \tag{42}$$

$$M_1 = \int_\omega x_1hf_3 + \int_{\gamma^+\cup\gamma_1^-\cup\gamma_3^-} x_1hg_3 + \int_{\gamma_2^-} x_1g_3, \tag{43}$$

$$M_2 = \int_\omega x_2h^2f_3 + \int_{\gamma^+\cup\gamma_1^-\cup\gamma_3^-} x_2h^2g_3 + \int_{\gamma_2^-} x_2hg_3. \tag{44}$$

Moreover, the axial stress component $\sigma^0_{33} \in L^2(\Omega)$ is given by:

$$\sigma^0_{33}(x_1,x_2,x_3) = E\big[\xi_3'(x_3) - x_1\xi_1''(x_3) - x_2h(x_3)\xi_2''(x_3)\big]. \tag{45}$$

Proof. From (24) and (25) we obtain that $e^*_{\alpha3}(u^0) = e^*_{\alpha\beta}(u^0) = 0$. Thus, by Lemma 1, we deduce the existence of $\xi_\alpha \in H^2_0(0,L)$ and $\xi_3 \in H^1_0(0,L)$ such that:

$$u^0_\alpha = \xi_\alpha, \qquad u^0_3 = \xi_3 - x_1\xi_1' - x_2h\xi_2'. \tag{46}$$

From (23) we obtain:

$$\frac{1}{E}\sigma^0_{33} - e^*_{33}(u^0) = 0, \tag{47}$$

which, combined with (46), allows us to conclude expression (45) for σ^0_{33}.

Taking in (27) $v_3 \in H_0^1(0,L)$ as a test function we obtain Eq. (41) in a direct way. Taking into account that, due to the symmetry, $\int_\omega x_1 h = \int_\omega x_1 x_2 h^2 = 0$, if we take now in (26) and in (27), respectively, $(v_\alpha) \in [H_0^2(0,L)]^2$ and $v_3 = x_1 v_1' + x_2 h v_2'$ as test functions we obtain the equation:

$$
\begin{aligned}
&-\int_0^L E\Big(\int_\omega x_1^2 h\Big)\xi_1'' v_1'' + \int_0^L E\Big[\Big(\int_\omega x_2 h^2\Big)\xi_3' - \Big(\int_\omega x_2^2 h^3\Big)\xi_2''\Big]v_2'' \\
&\quad = \int_0^L M_\alpha v_\alpha' - \int_0^L F_\alpha v_\alpha, \quad \forall (v_\alpha) \in [H_0^2(0,L)]^2
\end{aligned}
$$

which is equivalent to (39), (40). □

Remark 5. Note that, due to our selection of the reference cross-section ω, we have:

$$
\begin{aligned}
&\int_\omega h = |\omega^+| + H|\omega^-| = 1 + H, \\
&\int_\omega x_1^2 h = \int_{\omega+} x_1^2 + H\int_{\omega-} x_1^2 = \frac{1}{3} + \frac{1}{12}H, \\
&\int_\omega x_2 h^2 = \int_{\omega+} x_2 + H^2\int_{\omega-} x_2 = \frac{1}{4} - \frac{1}{2}H^2, \\
&\int_\omega x_2^2 h^3 = \int_{\omega+} x_2^2 + H^3\int_{\omega-} x_2^2 = \frac{1}{12} + \frac{1}{3}H^3.
\end{aligned}
$$

However, in the following we will use the general expression, which makes all equations valid for any cross-section.

In order to obtain a result of existence and uniqueness for the limit problem we will proof the following technical result:

Lemma 2. *The symmetric matrix:*

$$
A(x_3) = \begin{pmatrix} \int_\omega x_2^2 h^3 & -\int_\omega x_2 h^2 \\ -\int_\omega x_2 h^2 & \int_\omega h \end{pmatrix}
$$

is positive definite for all $x_3 \in [0,L]$.

Proof. By previous remark and applying that:

$$
\int_{\omega+} x_2^2 > 0, \qquad \int_{\omega-} x_2^2 > 0,
$$

it is immediate that:

$$
A_{11}(x_3) = \int_\omega x_2^2 h^3 \geqslant \delta > 0, \quad \forall x_3 \in [0,L]. \tag{48}
$$

On the other hand, the determinant of matrix $A(x_3)$ is given by:

$$D(x_3) = \Big(\int_\omega x_2^2 h^3\Big)\Big(\int_\omega h\Big) - \Big(\int_\omega x_2 h^2\Big)^2 = H^4(x_3)\Big[\Big(\int_{\omega^-} x_2^2\Big)\Big(\int_{\omega^-} 1\Big) - \Big(\int_{\omega^-} x_2\Big)^2\Big]$$
$$+ H^3(x_3)\Big(\int_{\omega^-} x_2^2\Big)\Big(\int_{\omega^+} 1\Big) - 2H^2(x_3)\Big(\int_{\omega^-} x_2\Big)\Big(\int_{\omega^+} x_2\Big)$$
$$+ H(x_3)\Big(\int_{\omega^-} 1\Big)\Big(\int_{\omega^+} x_2^2\Big) + \Big[\Big(\int_{\omega^+} x_2^2\Big)\Big(\int_{\omega^+} 1\Big) - \Big(\int_{\omega^+} x_2\Big)^2\Big].$$

Since

$$\Big(\int_{\omega^+} x_2\Big)^2 \leqslant \Big(\int_{\omega^+} x_2^2\Big)\Big(\int_{\omega^+} 1\Big), \qquad \Big(\int_{\omega^-} x_2\Big)^2 \leqslant \Big(\int_{\omega^-} x_2^2\Big)\Big(\int_{\omega^-} 1\Big),$$
$$\int_{\omega^+} 1 = |\omega^+| > 0, \qquad \int_{\omega^-} 1 = |\omega^-| > 0,$$
$$\int_{\omega^+} x_2 > 0, \qquad \int_{\omega^-} x_2 < 0,$$

and due to the boundedness (1) of H, we have that:

$$0 < \beta \leqslant D(x_3) \leqslant \gamma, \quad \forall x_3 \in [0, L]. \tag{49}$$

Thus, we obtain the positive definiteness of matrix $A(x_3)$. □

Then, as a consequence of previous lemma, we can prove the following result:

Theorem 2. *The limit problem* (39)–(41) *admits a unique solution* (ξ_i) *in the space* $[H_0^2(0, L)]^2 \times H_0^1(0, L)$. *Moreover, it is equivalent to the following differential problem*:

$$E\Big[\Big(\int_\omega x_1^2 h\Big)\xi_1''\Big]'' = F_1 + M_1' \quad in\ (0, L), \tag{50}$$

$$E\Big[\Big(\int_\omega x_2^2 h^3\Big)\xi_2'' - \Big(\int_\omega x_2 h^2\Big)\xi_3'\Big]'' = F_2 + M_2' \quad in\ (0, L), \tag{51}$$

$$E\Big[\Big(\int_\omega x_2 h^2\Big)\xi_2'' - \Big(\int_\omega h\Big)\xi_3'\Big]' = F_3 \quad in\ (0, L), \tag{52}$$

$$\xi_i(0) = \xi_i(L) = \xi_\alpha'(0) = \xi_\alpha'(L) = 0. \tag{53}$$

Proof. Since $\int_\omega x_1^2 h > 0$, Eq. (39) has a unique solution $\xi_1 \in H_0^2(0, L)$. Otherwise, the coupled system (40), (41) can be written in the equivalent way:

$$\hat{\xi} = \begin{pmatrix}\xi_2\\ \xi_3\end{pmatrix} \in H_0^2(0, L) \times H_0^1(0, L):$$
$$E\int_0^L (\xi_2''\ \xi_3')A(x_3)\begin{pmatrix}v_2''\\ v_3'\end{pmatrix} = \int_0^L (F_2 + M_2'\ F_3)\begin{pmatrix}v_2\\ v_3\end{pmatrix}, \quad \forall\hat{v} = \begin{pmatrix}v_2\\ v_3\end{pmatrix} \in H_0^2(0, L) \times H_0^1(0, L).$$

The bilinear form is $[H_0^2(0,L) \times H_0^1(0,L)]$-elliptic, due to the positive definiteness of the matrix $A(x_3)$ (Lemma 2). Thus, as a consequence of the Lax–Milgram theorem, the problem has a unique solution $(\xi_2, \xi_3) \in H_0^2(0,L) \times H_0^1(0,L)$. □

Remark 6. The problem (39)–(41) is uncoupled only in the case when H is constant. In this simple case we recover, as it was expected, the classical Bernoulli–Navier model for a constant cross-section rod.

Finally, we obtain the following convergence result, whose proof is similar to the ones in Bermúdez and Viaño [5] or Trabucho and Viaño [11] (see also Le Dret [10]):

Theorem 3. *If the system of applied forces verifies:*

$$f_\alpha \in L^2(\Omega), \quad g_\alpha \in L^2(\Gamma), \quad f_3 \in H^1(0,L;L^2(\omega)), \quad g_3 \in H^1(0,L;L^2(\gamma))$$

then we have the following convergences as $\varepsilon \to 0$:

$$u(\varepsilon) \to u^0 \quad \text{in } V(\Omega), \tag{54}$$
$$\sigma_{33}(\varepsilon) \to \sigma_{33}^0 \quad \text{in } L^2(\Omega), \tag{55}$$
$$\varepsilon\sigma_{\alpha 3}(\varepsilon) \to 0 \quad \text{in } L^2(\Omega), \tag{56}$$
$$\varepsilon^2\sigma_{\alpha\beta}(\varepsilon) \to 0 \quad \text{in } L^2(\Omega). \tag{57}$$

Introducing now the expressions only depending on x_3:

$$A_1 = \int_\omega x_1^2 x_2 h h', \qquad A_2 = \int_\omega x_1 x_2^2 h^2 h', \qquad J = -\int_\omega x_\alpha h \partial_\alpha \Psi,$$
$$I_1^w = \int_\omega x_1 h w, \qquad I_2^w = \int_\omega x_2 h^2 w, \qquad I_3^w = \int_\omega h w,$$
$$\tilde{I}_1^w = \int_\omega x_1 h' w, \qquad \tilde{I}_2^w = \int_\omega 2 x_2 h h' w, \qquad \tilde{I}_3^w = \int_\omega h' w,$$
$$I_1^\Psi = -\int_\omega x_2^2 h^2 \partial_2 \Psi, \qquad I_2^\Psi = \int_\omega x_1^2 h \partial_1 \Psi,$$
$$\tilde{I}_1^\Psi = -\int_\omega x_1 x_2 h h' \partial_1 \Psi, \qquad \tilde{I}_2^\Psi = -\int_\omega x_2^2 h^2 h' \partial_1 \Psi,$$

the auxiliary functions:

$$\Phi_{11}(x_1,x_2,x_3) = \frac{1}{2}(x_1^2 - x_2^2 h^2) = -\Phi_{22}(x_1,x_2,x_3),$$
$$\Phi_{12}(x_1,x_2,x_3) = x_1 x_2 h = \Phi_{21}(x_1,x_2,x_3),$$
$$\Phi_3(x_1,x_2,x_3) = \frac{1}{2}(x_1^2 + x_2^2 h^2) - \frac{1}{2}\int_\omega (x_1^2 + x_2^2 h^2),$$

the new Timoshenko's functions $\eta_\beta(x_1, x_2, x_3) = \eta_\beta(x_3)(x_1, x_2)$ solution of:

$$
\begin{aligned}
&-h\partial_{11}\eta_1 - h^{-1}\partial_{22}\eta_1 = -2x_1 h \quad \text{in } \omega \times (0, L),\\
&-h\partial_{11}\eta_2 - h^{-1}\partial_{22}\eta_2 = -2\left[x_2 h^2 - \left(\int_\omega x_2 h^2\right)\right] \quad \text{in } \omega \times (0, L),\\
&h\partial_1\eta_\beta n_1 + h^{-1}\partial_2\eta_\beta n_2 = 0 \quad \text{in } \gamma \times (0, L),\\
&\int_\omega \eta_\beta = 0 \quad \text{in } (0, L),
\end{aligned}
$$

the functions $\theta_\beta(x_1, x_2, x_3) = \theta_\beta(x_3)(x_1, x_2)$ solution of:

$$
\begin{aligned}
&-h\partial_1(\partial_1\theta_\beta + \Phi_{\beta 1}) - h^{-1}\partial_2(\partial_2\theta_\beta + h\Phi_{\beta 2}) = 0 \quad \text{in } \omega \times (0, L),\\
&h(\partial_1\theta_\beta + \Phi_{\beta 1})n_1 + h^{-1}(\partial_2\theta_\beta + h\Phi_{\beta 2})n_2 = 0 \quad \text{in } \gamma \times (0, L),\\
&\int_\omega \theta_\beta = 0 \quad \text{in } (0, L),
\end{aligned}
$$

and the functions $\lambda_\beta(x_1, x_2, x_3) = \lambda_\beta(x_3)(x_1, x_2)$ solution of:

$$
\begin{aligned}
&-h\partial_{11}\lambda_1 - h^{-1}\partial_{22}\lambda_1 = -x_1 h' \quad \text{in } \omega \times (0, L),\\
&-h\partial_{11}\lambda_2 - h^{-1}\partial_{22}\lambda_2 = -\left[x_2 h^2 - \left(\int_\omega x_2 h^2\right)\right]' \quad \text{in } \omega \times (0, L),\\
&h\partial_1\lambda_1 n_1 + h^{-1}\partial_2\lambda_1 n_2 = -x_1 x_2 h' n_2 \quad \text{in } \gamma \times (0, L),\\
&h\partial_1\lambda_2 n_1 + h^{-1}\partial_2\lambda_2 n_2 = -x_2^2 h h' n_2 \quad \text{in } \gamma \times (0, L),\\
&\int_\omega \lambda_\beta = 0 \quad \text{in } (0, L),
\end{aligned}
$$

we obtain the following characterization of u^2 and $\sigma^0_{i\alpha}$. We note that, in general, $u^2 \notin V(\Omega)$ due to a boundary layer phenomenon. This result provides us with a torsion model including warping effects, and its proof (which is not included here because of its extreme tediousness) is similar to the one given by Trabucho and Viaño [11] for the case of classical beams:

Theorem 4. *If the system of applied forces verifies*:

$$
f_\alpha \in L^2(\Omega), \quad g_\alpha \in L^2(\Gamma), \quad f_3 \in H^1(0, L; L^2(\omega)), \quad g_3 \in H^1(0, L; L^2(\gamma))
$$

then:

(a) *The limit displacement* $u^2 \in [H^1(\Omega)]^3$ *is of the form*:

$$
u_1^2 = z_1 + x_2 h z - \nu[x_1\xi_3' - \Phi_{1\beta}\xi_\beta''], \tag{58}
$$

$$
u_2^2 = z_2 - x_1 z - \nu[x_2 h\xi_3' - \Phi_{2\beta}\xi_\beta''], \tag{59}
$$

$$
u_3^2 = z_3 - x_1 z_1' - x_2 h z_2' + U_3, \tag{60}
$$

with:

$$U_3 = -wz' + [\nu\Phi_3 + (1+\nu)\widetilde{\Phi}_3]\xi_3'' + (1+\nu)(\Phi_3' + \widehat{\Phi}_3)\xi_3'$$
$$+ [(1+\nu)\eta_\alpha + \nu\theta_\alpha]\xi_\alpha''' + 2(1+\nu)\lambda_\alpha\xi_\alpha'' + \frac{2(1+\nu)}{E}w^0.$$

(b) *The shear stress components* $\sigma_{3\alpha}^0 \in L^2(\Omega)$ *are given by*:

$$\sigma_{31}^0 = \frac{E}{2(1+\nu)}\{-h^{-1}\partial_2\Psi z' + (1+\nu)\partial_1\widetilde{\Phi}_3\xi_3'' + (1+\nu)\partial_1\widehat{\Phi}_3\xi_3'$$
$$+ [(1+\nu)\partial_1\eta_\alpha + \nu\partial_1\theta_\alpha + \nu\Phi_{1\alpha}]\xi_\alpha''' + 2(1+\nu)\partial_1\lambda_\alpha\xi_\alpha''\} + \partial_1 w^0, \tag{61}$$

$$\sigma_{32}^0 = \frac{E}{2(1+\nu)}\{\partial_1\Psi z' + (1+\nu)h^{-1}\partial_2\widetilde{\Phi}_3\xi_3'' + (1+\nu)[h^{-1}\partial_2\widehat{\Phi}_3 + 2x_2h']\xi_3' \tag{62}$$
$$+ [(1+\nu)h^{-1}\partial_2\eta_\alpha + \nu h^{-1}\partial_2\theta_\alpha + \nu\Phi_{2\alpha}]\xi_\alpha''' + 2(1+\nu)h^{-1}\partial_2\lambda_\alpha\xi_\alpha''\} + h^{-1}\partial_2 w^0.$$

(c) *The plane stress components* $\sigma_{\alpha\beta}^0 \in L^2(\Omega)$ *takes the form*:

$$\sigma_{\alpha\beta}^0 = \sigma_{\alpha\beta}^*(\underline{u}) + \underline{\sigma}_{\alpha\beta}, \tag{63}$$

with:

$$\sigma_{\alpha\beta}^*(\underline{u}) = \frac{E}{1+\nu}e_{\alpha\beta}^*(\underline{u}) + \frac{\nu E}{(1+\nu)(1-2\nu)}e_{\mu\mu}^*(\underline{u})\delta_{\alpha\beta},$$
$$\underline{\sigma}_{\alpha\beta} = \frac{\nu E.}{(1+\nu)(1-2\nu)}(z_3' - x_1z_1'' - x_2hz_2'' + U_3' - x_2h'h^{-1}\partial_2U_3)\delta_{\alpha\beta},$$

where:

1) *The warping function* $w(x_1,x_2,x_3) = w(x_3)(x_1,x_2)$ *is the unique solution of*:

$$w \in H^1[0,L;H^1(\omega)]:$$
$$-h\partial_{11}w - h^{-1}\partial_{22}w = 0 \quad in\ \omega\times(0,L), \tag{64}$$
$$h\partial_1 wn_1 + h^{-1}\partial_2 wn_2 = x_2h^2n_1 - x_1n_2 \quad in\ \gamma\times(0,L), \tag{65}$$
$$\int_\omega w = 0 \quad in\ (0,L). \tag{66}$$

2) *The torsion function* $\Psi(x_1,x_2,x_3) = \Psi(x_3)(x_1,x_2)$ *is the unique solution of*:

$$\Psi \in H^1[0,L;H_0^1(\omega)]:$$
$$-h\partial_{11}\Psi - h^{-1}\partial_{22}\Psi = 2h \quad in\ \omega\times(0,L). \tag{67}$$

3) *The additional warping function* $w^0(x_1, x_2, x_3) = w^0(x_3)(x_1, x_2)$ *is the unique solution of:*

$$w^0 \in H^1[0, L; H^1(\omega)]:$$

$$-h\partial_{11}w^0 - h^{-1}\partial_{22}w^0 = hf_3 - F_3 \quad \textit{in } \omega \times (0, L), \tag{68}$$

$$h\partial_1 w^0 n_1 + h^{-1}\partial_2 w^0 n_2 = hg_3 \quad \textit{in } (\gamma^+ \cup \gamma_1^- \cup \gamma_3^-) \times (0, L), \tag{69}$$

$$h\partial_1 w^0 n_1 + h^{-1}\partial_2 w^0 n_2 = g_3 \quad \textit{in } \gamma_2^- \times (0, L), \tag{70}$$

$$\int_\omega w^0 = 0 \quad \textit{in } (0, L). \tag{71}$$

4) *The new functions* $\widetilde{\Phi}_3(x_1, x_2, x_3) = \widetilde{\Phi}_3(x_3)(x_1, x_2)$ *and* $\widehat{\Phi}_3(x_1, x_2, x_3) = \widehat{\Phi}_3(x_3)(x_1, x_2)$ *are, respectively, the unique solution of:*

$$\widetilde{\Phi}_3 \in H^1[0, L; H^1(\omega)]:$$

$$-h\partial_{11}\widetilde{\Phi}_3 - h^{-1}\partial_{22}\widetilde{\Phi}_3 = 2\left[h - \left(\int_\omega h\right)\right] \quad \textit{in } \omega \times (0, L), \tag{72}$$

$$h\partial_1\widetilde{\Phi}_3 n_1 + h^{-1}\partial_2\widetilde{\Phi}_3 n_2 = 0 \quad \textit{in } \gamma \times (0, L), \tag{73}$$

$$\int_\omega \widetilde{\Phi}_3 = 0 \quad \textit{in } (0, L). \tag{74}$$

$$\widehat{\Phi}_3 \in H^1[0, L; H^1(\omega)]:$$

$$-h\partial_{11}\widehat{\Phi}_3 - h^{-1}\partial_{22}\widehat{\Phi}_3 = 2\left[h - \left(\int_\omega h\right)\right]' \quad \textit{in } \omega \times (0, L), \tag{75}$$

$$h\partial_1\widehat{\Phi}_3 n_1 + h^{-1}\partial_2\widehat{\Phi}_3 n_2 = 0 \quad \textit{in } \gamma \times (0, L), \tag{76}$$

$$\int_\omega \widehat{\Phi}_3 = 0 \quad \textit{in } (0, L). \tag{77}$$

5) *The twist angle* $z(x_3)$ *is solution of the problem:*

$$z \in H^1(0, L):$$

$$\int_0^L \frac{E}{2(1+\nu)} J z'\xi' = \int_0^L M_3\xi + \int_0^L M_w\xi', \quad \forall \xi \in H_0^1(0, L), \tag{78}$$

with:

$$M_3 = \int_\omega h(x_2 h f_1 - x_1 f_2) + \int_{\gamma^+\cup\gamma_1^-\cup\gamma_3^-} h(x_2 h g_1 - x_1 g_2) + \int_{\gamma_2^-} (x_2 h g_1 - x_1 g_2),$$

$$M_w = -\int_\omega h f_3 w - \int_{\gamma^+\cup\gamma_1^-\cup\gamma_3^-} h g_3 w - \int_{\gamma_2^-} g_3 w$$
$$+ E\left\{\left[I_\alpha^w + \frac{\nu}{2(1+\nu)} I_\alpha^\Psi\right]\xi_\alpha''' + (\widetilde{I}_\alpha^w + \widetilde{I}_\alpha^\Psi - A_\alpha)\xi_\alpha'' - I_3^w \xi_3'' - \widetilde{I}_3^w \xi_3'\right\}.$$

6) *The second order bending $z_\alpha(x_3)$ and the second order stretching $z_3(x_3)$ are solution of the coupled problem*:

$$z_\alpha \in H^2(0,L), \quad z_3 \in H^1(0,L):$$

$$-\int_0^L E\Big(\int_\omega x_1^2 h\Big) z_1'' v_1'' = \int_0^L G_1 v_1'' + \int_0^L \widetilde{M}_1 v_1' - \int_0^L \widetilde{F}_1 v_1, \quad \forall v_1 \in H_0^2(0,L), \tag{79}$$

$$\int_0^L E\Big[\Big(\int_\omega x_2 h^2\Big) z_3' - \Big(\int_\omega x_2^2 h^3\Big) z_2''\Big] v_2'' = \int_0^L G_2 v_2'' + \int_0^L \widetilde{M}_2 v_2' - \int_0^L \widetilde{F}_2 v_2, \quad \forall v_2 \in H_0^2(0,L), \tag{80}$$

$$\int_0^L E\Big[\Big(\int_\omega h\Big) z_3' - \Big(\int_\omega x_2 h^2\Big) z_2''\Big] v_3' = \int_0^L G_3 v_3' + \int_0^L \widetilde{F}_3 v_3, \quad \forall v_3 \in H_0^1(0,L), \tag{81}$$

with:

$$\widetilde{F}_i = \frac{1}{2}\int_{\gamma_2^-} (h')^2 g_i,$$

$$\widetilde{M}_1 = \frac{1}{2}\int_{\gamma_2^-} x_1 (h')^2 g_3, \qquad \widetilde{M}_2 = \frac{1}{2}\int_{\gamma_2^-} x_2 h (h')^2 g_3,$$

$$G_1 = \int_\omega x_1 h\{E(U_3' - h'h^{-1}x_2\partial_2 U_3) + \nu\sigma_{\alpha\alpha}^0\},$$

$$G_2 = \int_\omega x_2 h^2\{E(U_3' - h'h^{-1}x_2\partial_2 U_3) + \nu\sigma_{\alpha\alpha}^0\},$$

$$G_3 = -\int_\omega h\{E(U_3' - h'h^{-1}x_2\partial_2 U_3) + \nu\sigma_{\alpha\alpha}^0\}.$$

7) *The fourth order displacement $\underline{u} \equiv (\underline{u}_\alpha(x_3)(x_1,x_2))$ is the unique solution of*:

$$\underline{u} \in L^2(0,L;[H^1(\omega)]^2):$$

$$\int_\omega h\sigma_{\alpha\beta}^*(\underline{u})e_{\alpha\beta}^*(\phi) = \int_\omega hf_\alpha\phi_\alpha + \int_{\gamma^+\cup\gamma_1^-\cup\gamma_3^-} hg_\alpha\phi_\alpha + \int_{\gamma_2^-} g_\alpha\phi_\alpha + \int_\omega \partial_3(h\sigma_{\alpha3}^0)\phi_\alpha + \int_\omega x_2 h'\partial_2\phi_\alpha\sigma_{\alpha3}^0 - \int_\omega h\underline{\sigma}_{\alpha\beta}e_{\alpha\beta}^*(\phi), \quad \forall\phi \in [H^1(\omega)]^2, \tag{82}$$

$$\int_\omega \underline{u}_\alpha = \int_\omega (x_2 h\underline{u}_1 - x_1\underline{u}_2) = 0 \quad \textit{in } (0,L). \tag{83}$$

Remark 7. Although we have presented here the study of beams with longitudinally variable depth and simply connected cross-section, the previous model can be extended without difficulties to the case of a multiply connected cross-section, similar to the one represented in Fig. 3. The only differences in this case are the additional boundary conditions on the interior boundary of ω for the problems corresponding to the characterization of the different terms appearing in previous theorems.

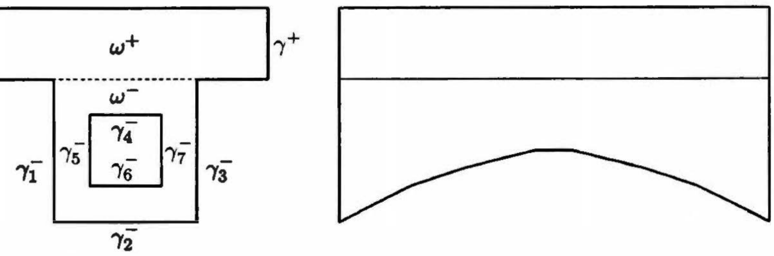

Fig. 3. Typical beam with multiply connected section ω.

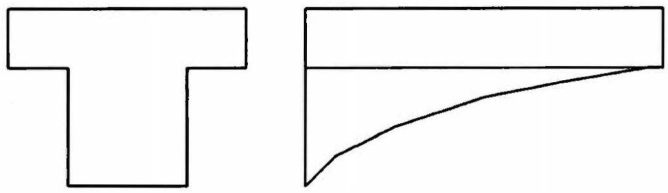

Fig. 4. Cantilever beam.

In the same way, this new model can be also extended to the evolutive and the nonlinear cases by following the techniques introduced by Alvarez et al. [2,3] for classical beams.

Finally, we must remark again that the model can be extended in a straight-forward way to other types of boundary conditions (simply supported or hinged ends, cantilever beams, ...).

Remark 8. In the case where hypothesis (1) is not verified, that is, when the "shape" function H approaches zero, we can also obtain a limit model considering a sequence of "shape" functions $\{H_\alpha\}_{\alpha>0}$ verifying (1) and converging to H in $L^\infty(0, L)$ as α goes to zero, and then taking limits in the corresponding models. In this case, expressions where h^{-1} is present must be used in the equivalent form obtained by multiplying by h. So, for instance, Eq. (67) should be used in the equivalent form:

$$-h^2\partial_{11}\Psi - \partial_{22}\Psi = 2h^2 \quad \text{in } \omega \times (0, L).$$

This fact allows us to consider, for instance, a cantilever longitudinally variable cross-section beam with a top constant slab (as given in Fig. 4), corresponding to a "shape" function:

$$H(x_3) = \left(1 - \frac{x_3}{L}\right)^2,$$

which is zero a point $x_3 = L$. The results obtained from the present analysis can be compared to the approximate ones found in the current engineering practice by using the shear stress formulae including the Resal effect.

4. Typical example of longitudinally variable cross-section beam

We present here a classical example of longitudinally variable, multiply connected cross-section beam, usually employed in civil engineering (bridges and buildings frame structures).

It can be seen in Fig. 3, that this beam corresponds to a "parabolic" bridge with "shape" function:

$$H(x_3) = 1 - 2\left\{\frac{x_3}{L} - \left(\frac{x_3}{L}\right)^2\right\},$$

and whose rescaled cross-section is given by the multiply connected domain $\omega = \omega^+ \cup \omega^-$, where

$$\omega^+ = (-1,1) \times \left(0, \frac{1}{2}\right), \qquad \omega^- = \left(-\frac{1}{2}, \frac{1}{2}\right) \times (-1,0] \backslash \left[-\frac{1}{4}, \frac{1}{4}\right] \times \left[-\frac{3}{4}, -\frac{1}{4}\right].$$

For $\Omega = \omega \times (0,L)$, in addition to the previous parts of the boundary Γ_0, Γ^+, Γ_1^-, Γ_2^- and Γ_3^-, we also consider the new parts $\Gamma_i^- = \gamma_i^- \times (0,L)$, $i = 4,\dots,7$, with $\bigcup_{i=4}^7 \gamma_i^-$ the interior boundary of ω^- (cf. Fig. 3). Finally, we redefine the lateral boundary:

$$\Gamma = \gamma \times (0,L) \quad \text{with } \gamma = \bigcup_{i=1}^{7} \gamma_i^- \cup \gamma^+.$$

Then, making the computations for this particular case, we obtain that the first-order displacement u^0 is the generalized Bernoulli–Navier displacement:

$$\begin{aligned}
&u_\alpha^0 = \xi_\alpha, \\
&u_3^0 = \xi_3 - x_1\xi_1' - x_2\xi_2' \qquad \text{for } x_2 \geqslant 0, \\
&u_3^0 = \xi_3 - x_1\xi_1' - x_2 H\xi_2' \quad \text{for } x_2 \leqslant 0,
\end{aligned}$$

where $(\xi_i) \in [H_0^2(0,L)]^2 \times H_0^1(0,L)$ is the unique solution of the coupled problem:

$$\begin{aligned}
&E\left[\left(\frac{1}{3} + \frac{5}{64}H\right)\xi_1''\right]'' = F_1 + M_1' \quad \text{in } (0,L), \\
&E\left[\left(\frac{1}{12} + \frac{17}{64}H^3\right)\xi_2'' - \left(\frac{1}{4} - \frac{3}{8}H^2\right)\xi_3'\right]'' = F_2 + M_2' \quad \text{in } (0,L), \\
&E\left[\left(\frac{1}{4} - \frac{3}{8}H^2\right)\xi_2'' - \left(1 + \frac{3}{4}H\right)\xi_3'\right]' = F_3 \quad \text{in } (0,L),
\end{aligned}$$

with:

$$\begin{aligned}
&F_i = \int_{\omega^+} f_i + H\int_{\omega^-} f_i + \int_{\gamma^+\cup\gamma_2^-\cup\gamma_4^-\cup\gamma_6^-} g_i + H\int_{\gamma_1^-\cup\gamma_3^-\cup\gamma_5^-\cup\gamma_7^-} g_i, \\
&M_1 = \int_{\omega^+} x_1 f_3 + H\int_{\omega^-} x_1 f_3 + \int_{\gamma^+\cup\gamma_2^-\cup\gamma_4^-\cup\gamma_6^-} x_1 g_3 + H\int_{\gamma_1^-\cup\gamma_3^-\cup\gamma_5^-\cup\gamma_7^-} x_1 g_3, \\
&M_2 = \int_{\omega^+} x_2 f_3 + H^2\int_{\omega^-} x_2 f_3 + \int_{\gamma^+} x_2 g_3 + H\int_{\gamma_2^-\cup\gamma_4^-\cup\gamma_6^-} x_2 g_3 + H^2\int_{\gamma_1^-\cup\gamma_3^-\cup\gamma_5^-\cup\gamma_7^-} x_2 g_3.
\end{aligned}$$

Moreover, the first-order shear stress component $\sigma_{33}^0 \in L^2(\Omega)$ is given by the expressions:

$$\sigma_{33}^0 = E\{\xi_3' - x_1\xi_1'' - x_2\xi_2''\} \qquad \text{for } x_2 \geqslant 0,$$
$$\sigma_{33}^0 = E\{\xi_3' - x_1\xi_1'' - x_2 H\xi_2''\} \quad \text{for } x_2 \leqslant 0.$$

Following the results of Theorem 4, we can compute in the same way the corresponding second-order displacement u^2 and the other components of the first-order stress field, which can be compared to the classical results (see Fernández et al. [8,9] for straight beams).

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